Fall 2013 **Math 601** Name: ____________________

Sample Quiz 10

Question 1. (10 pts)

Evaluate the integral

$$\int_C \frac{e^{2z}}{z-\pi i}dz$$

(a) when C is the circle $|z-1|=4$, that is, the circle centered at 1 with radius 4.

> **Solution:** Notice that the distance between 1 and πi is
>
> $$|1-\pi i| = \sqrt{1+\pi^2} < 4.$$
>
> So πi lies inside the circle $|z-1|=4$. So apply Cauchy's formula
>
> $$\frac{1}{2\pi i}\int_C \frac{e^{2z}}{z-\pi i}dz = e^{2\pi i} = 1$$
>
> Therefore
>
> $$\int_C \frac{e^{2z}}{z-\pi i}dz = 2\pi i$$

(b) when C is the unit circle $|z|=1$, i.e., the circle centered at 0 with radius 1.

> **Solution:** Clearly, πi is outside the circle $|z|=1$. Then the function $f(z) = \frac{e^{2z}}{z-\pi i}$ is analytic on the disk $|z|\le 1$. So apply Cauchy's theorem,
>
> $$\frac{1}{2\pi i}\int_C \frac{e^{2z}}{z-\pi i}dz = 0$$
>
> Therefore
>
> $$\int_C \frac{e^{2z}}{z-\pi i}dz = 0$$

Question 2. (10 pts)

Show that

$$\frac{1}{2\pi i}\int_C \frac{e^{zt}}{z^2+1}dz = \sin t$$

where C is the the circle $|z| = 3$.

Solution:

$$\frac{e^{zt}}{z^2+1} = \frac{e^{zt}}{(z+i)(z-i)}$$

Both i and $-i$ lie inside the circle $|z| = 3$. Consider the partial fractions

$$\frac{1}{(z+i)(z-i)} = \frac{A}{z-i} - \frac{B}{z+i} = \frac{A(z+i) - B(z-i)}{(z+i)(z-i)}$$

So $A = B = \frac{1}{2i}$. Now apply Cauchy's formula

$$\begin{aligned}
&\frac{1}{2\pi i}\int_C \frac{e^{zt}}{z^2+1}dz \\
&= \frac{1}{2\pi i}\int_C \frac{e^{zt}}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)dz \\
&= \frac{e^{it}}{2i} - \frac{e^{-it}}{2i} \\
&= \frac{\cos t + i\sin t}{2i} - \frac{\cos t - i\sin t}{2i} \\
&= \sin t
\end{aligned}$$